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JHEP08(2009)073

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On half-BPS states of the ABJM theory

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ABSTRACT: We analyze SU(2) invariant half-BPS states of the 3d, $\mathcal{N} = 8$ or $\mathcal{N} = 6$ SCFT within the radial quantization of the ABJM theory [1], the theory proposed to describe N M2-branes in the $R^3 \times \mathbb{C}^4/Z_k$ background. After studying the classical moduli space of these configurations, we explicitly construct a set of gauge invariant operators involving 't Hooft monopole operators corresponding to these states. We show there is a one-to-one correspondence between the two sets carrying R-charge J and that they are labeled by Young tableaux of J boxes with a maximum of N rows. Restricting the full path integral to this half-BPS sector of the theory, we show the latter is described in terms of N fermions in a 2d harmonic potential in the sector of vanishing angular momentum. The same classification, though in the $N \rightarrow \infty$ limit, arise from the plane-wave (BMN) Matrix theory as well as the 11 dimensional LLM bubbling geometries [2], providing supportive evidence for the ABJM theory and/or the Matrix model.

KEYWORDS: AdS-CFT Correspondence, M-Theory, M(atrrix) Theories

ARXIV EPRINT: [0904.4605](https://arxiv.org/abs/0904.4605)

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1 Introduction

The $\mathcal{N} = 8$ 3d SCFT is the theory describing the low energy limit of multiple M2-branes in $R^{1,10}$. Moreover, it is expected to be the conformal field theory dual to M-theory on the $AdS_4 \times S^7$ background. As such the recent proposals for a non-trivial interacting $\mathcal{N} = 8$ 3d field theory [3], the BLG theory, has prompted an extensive study of these models. The original BLG theory, with totally antisymmetric three-bracket structure [3] and a three-algebra with positive definite metric seems to be well suited for describing dynamics of two M2-branes [4]. The way to construct a theory describing a generic number of M2-branes was paved by the realization that the BLG theory may also be written as a $SU(2) \times SU(2)$ 3d (supersymmetric) Chern-Simons gauge theory with the $SU(2)$'s to have levels k and $-k$ [5].

The generalization to N M2-branes, for which the natural guess would be an $SU(N) \times SU(N)$ supersymmetric Chern-Simons gauge theory, now known as the ABJM theory, was proposed in [1] shortly after the BLG theory. It was shown, through a construction involving N D3-brane intersecting an NS5-brane and a $(1, k)$ 5-brane of type IIB theory and uplifting the system to M-theory and taking the low-energy limit, that the theory describing N M2-branes probing a (supersymmetric) C^4/\mathbb{Z}_k orbifold is a supersymmetric $U(N)_k \times U(N)_{-k}$ Chern-Simons theory [1]. This theory, which is closely resembling the BLG theory for the $N = 2$, has $\mathcal{N} = 6$ supersymmetry, it is a conformal field theory and it is invariant under the $Osp(4^*|6)$ superalgebra [6]. For $k = 1, 2$ the ABJM theory is expected to become an $\mathcal{N} = 8$ 3d theory. This theory has passed many tests and many extensions of the model (to less supersymmetric Chern-Simons gauge theories) have also been studied.

In this note we study and classify all the half-BPS configurations of the ABJM theory which are invariant under the $SU(2|3)$ superalgebra and compare it with the known results

from the plane-wave matrix theory [7] and the half-BPS deformations of the eleven dimensional plane-wave background [2]. We show that there is a one-to-one correspondence between these three. The half-BPS states of an $\mathcal{N} = 8$ 3d theory are labeled by the only quantum number they carry J , which is the R-charge corresponding to a $U(1) \in SO(8)$ R-symmetry group of the theory. Being BPS the scaling dimension of these operators $\Delta = J/2$ is protected by supersymmetry.

To study the half-BPS states of the ABJM theory we need to consider monopole (or 't Hooft line) operators. In the radial quantization of the ABJM theory (i.e. the ABJM theory on $R \times S^2$) these are operators which involve a non-zero magnetic flux on the S^2 . Due to the presence of the 't Hooft monopole operator, these half-BPS operators may seem to be non-local. However, since we are dealing with a Chern-Simons theory their non-local part is a gauge artifact [1, 8].¹

As we will show the half-BPS states with R-charge J are constructed from monopole operators the total magnetic flux of which over the S^2 is J . Here we will give a detailed construction of half-BPS operators and their classification by all possibilities available for monopole operators of flux J . As we will discuss such monopole operators are classified by the partition of J into N non-negative integers (N is the rank of the gauge group in the corresponding ABJM theory). Therefore all the half-BPS states of the ABJM theory, for any k , are labeled by Young tableaux of J number of boxes and maximum N number of rows.

It is well established that (e.g. see [2, 10]) the half-BPS sector of the $\mathcal{N} = 4$ $SU(N)$ SYM is equivalent to a system of N $2d$ fermions in the Lowest Landau Level. In this work we show that a similar $2d$ fermionic picture is also true for the half-BPS sector of the ABJM theory (for any k). In this case, in contrast to the $\mathcal{N} = 4$ SYM case, the fermions are bound to move in a $2d$ harmonic oscillator potential in the sector with zero angular momentum.

The ABJM theory at level $k = 1, 2$ is a theory dual to M-theory on $AdS_4 \times S^7$ (or its Penrose or plane-wave limit, the $11d$ plane-wave) and as such one expects to have a similar classification for half-BPS states of the latter theory. Although an independent formulation for M-theory on $AdS_4 \times S^7$ is still lacking, for this purpose one can use the plane-wave matrix theory as the discrete light cone quantization (DLCQ) of M-theory on the $11d$ plane-wave background.² The half-BPS states of the latter have been classified and shown to be all labeled by the J dimensional representation of $SU(2)$ [12]. These representations are labeled by Young tableaux of J boxes. In contrast to the ABJM case, there is no restriction on the number of rows of the Young diagrams in this case.

As the third description for these half-BPS states, we consider the class of $11d$ supergravity solutions which are half-BPS deformations of $11d$ maximally supersymmetric plane-wave. These geometries are the $11d$ LLM (bubbling) geometries [2]. As discussed in [13] (see also [14, 15]) these geometries are labeled by a set of integers specifying the number of spherical M2 or M5-branes in the background. We show how this information can naturally be encoded in a Young tableau, in perfect agreement with the previous two descriptions.

¹For recent work on the construction of gauge invariant operators describing warped M2-branes see [9].

²The $11d$ plane-wave and the $AdS_4 \times S^7$ are related by taking the Penrose limit. The process of taking the Penrose limit closely resembles that of going to an infinite momentum frame and/or the DLCQ [11]. The plane-wave matrix model can also be considered as the DLCQ of M-theory on the $AdS_4 \times S^7$.

This paper is organized as follows. In section 2, after reviewing the ABJM theory we focus on its half-BPS sector and construct all the half-BPS operators in the sector with R-charge J . In section 3, we show that the half-BPS sector of the ABJM theory is described by a $2d$ fermion system. In section 4, we review the results of the plane-wave matrix theory and its half-BPS states. Moreover, we review the $11d$ LLM bubbling geometries and show that the half-BPS deformations of the $11d$ plane-wave can be described by a Young tableau, similarly to the one used for half-BPS states of the ABJM theory. In this way we give a natural interpretation for the monopole operators of the ABJM theory in terms of spherical M2 or M5 -brane giant gravitons. The last section is devoted to discussions.

2 Half-BPS sector in the ABJM theory

The ABJM theory is a supersymmetric $3d$ Chern-Simons theory with four complex scalars Z^A and four two component $3d$ fermions ψ_A in the $\mathbf{4}$ of the $SO(6)$ R-symmetry group. These matter fields transform in the bi-fundamental (N, \bar{N}) representation of $u(N) \times u(N)$. Besides these propagating fields, there are a couple of non-dynamical Chern-Simons gauge fields $A^{(1)}$ and $A^{(2)}$ in the $N \times N$ representations of each of the $u(N)$ algebras. They have a Chern-Simons action with opposite integer levels for the two gauge groups, k and $-k$.

These theories are dual to $AdS_4 \times S^7/Z_k$. For the particular values of $k = 1, 2$, the R-symmetry group is enhanced to $SO(8)$ and the number of supersymmetries to 32.

We are interested in studying the half-BPS sector of these theories preserving $SO(3) \times SU(4)$, for $k = 1, 2$ or $SO(3) \times SU(3)$ for $k \neq 1, 2$. These are states saturating the BPS bound $\Delta = J/2$, where J corresponds to their $U(1)$ R-charge and Δ to their conformal dimension. Since R-charge acts as rotation on the complex scalar fields, the preserved symmetries guarantee that such states will only involve a single scalar field $Z = Z^1$, carrying one unit of R-charge and having conformal dimension $1/2$.

It is convenient to use radial quantization by defining these theories on $R \times S^2$ so that there exists an operator-state correspondence. In addition, the eigenvalues of the Hamiltonian correspond to the scaling dimensions of the operators. The classical action will involve an extra mass coupling of the scalar fields to the curvature of the 2-sphere [1, 17, 18]. The bosonic truncation of the total action to this single complex scalar field Z coupled to the two gauge fields is

$$S = -\frac{1}{8\pi} \int dt d^2\Omega Tr \left[D_\alpha Z D^\alpha \bar{Z} + D_\alpha \bar{Z} D^\alpha Z - \frac{1}{4}(\bar{Z}Z + Z\bar{Z}) \right] - S_{CS}, \quad (2.1)$$

where S_{CS} stands for the Chern-Simons piece

$$S_{CS} = \frac{k}{8\pi} \int dt d^2\Omega Tr \sum_{i=1}^2 (-1)^{i+1} \left(A^{(i)} \wedge dA^{(i)} + \frac{2}{3} A^{(i)} \wedge A^{(i)} \wedge A^{(i)} \right), \quad (2.2)$$

and the covariant derivatives are defined according to the field representation,

$$\begin{aligned} D_\alpha Z &= \partial_\alpha Z + iA_\alpha^{(1)} Z - iZ A_\alpha^{(2)}, \\ D_\alpha \bar{Z} &= \partial_\alpha \bar{Z} + iA_\alpha^{(2)} \bar{Z} - i\bar{Z} A_\alpha^{(1)}. \end{aligned} \quad (2.3)$$

We have chosen the radius of the S^2 such that the mass is (formally) equal to one half.

2.1 Classical moduli space of half-BPS configurations

In the following, we will solve the classical equations of motion derived from (2.1) while preserving the appropriate bosonic symmetries. We will then check that these configurations preserve half of the supersymmetries.³

Classical half-BPS configurations must be SU(2) invariant. As such, the matrix Z must be covariantly constant on the 2-sphere, i.e. $D_\theta Z = D_\phi Z = 0$. Non-vanishing R-charge requires a non-trivial time dependence on Z to describe the rotation in the $\{Z, \bar{Z}\}$ subspace. Since (2.1) contains a Chern-Simons term, gauge fields cannot be turned off consistently. Their equations of motion are

$$\begin{aligned} \epsilon^{\mu\nu\alpha} F_{\mu\nu}^{(1)} &= \frac{2\pi i}{k} J^\alpha = \frac{2\pi i}{k} (\bar{Z} D^\alpha Z - (D^\alpha \bar{Z}) Z) , \\ \epsilon^{\mu\nu\alpha} F_{\mu\nu}^{(2)} &= -\frac{2\pi i}{k} \bar{J}^\alpha = -\frac{2\pi i}{k} (Z D^\alpha \bar{Z} - (D^\alpha Z) \bar{Z}) . \end{aligned} \tag{2.4}$$

Notice that $D_\theta Z = D_\phi Z = 0$ is on-shell equivalent to the absence of covariantly constant vector fields $F_{t\mu}^{(i)}$ on the 2-sphere. Since J^α is a conserved current, we learn that the $N \times N$ matrices

$$F^{(i)} \equiv \int_{S^2} F_{\theta\phi}^{(i)} \tag{2.5}$$

are constants of motion, that is $D_t F^{(i)} = 0$. Note that on a 2-sphere we have an invariant two-form, its volume-form, and hence $F_{\theta\phi} \propto \sin\theta$ is not ruled out by the SO(3) invariance condition.⁴

Let us next consider the equations of motion for the Z field which in the absence of the gauge fields take the form

$$-\partial_t^2 Z + \nabla^2 Z - \frac{1}{4} Z = 0.$$

One may use the $u(N) \times u(N)$ gauge transformations to diagonalize Z . Let us work in a gauge where

$$Z = \text{diag}(z_1, z_2, \dots, z_N), \quad z_i = e^{-i\omega_i t} w_i . \tag{2.6}$$

we learn that the spectrum of Z is

$$\omega_l = \sqrt{l(l+1) + \frac{1}{4}} = l + \frac{1}{2} .$$

Hence for the half-BPS sector, where all the fields should be constant on the S^2 (i.e. $l = 0$) ω_i are all equal to $\frac{1}{2}$ in units of the radius of the S^2 .

Working in the gauge $A^{(1)} = A^{(2)}$, conditions $D_\theta Z = D_\phi Z = 0$ are manifestly satisfied for Z 's in the half-BPS sector. This gauge is preserved by a diagonal $u(N)$, which can

³Our analysis is close in nature to the one presented in section III of [18]. Here, however, we directly focus on the subset of degrees of freedom relevant for the description of half-BPS states with the appropriate bosonic symmetries.

⁴This should be contrasted with the half-BPS sector of $\mathcal{N} = 4$ SYM on $R \times S^3$, in the sense that there are no invariant two-forms on the S^3 .

be used to set $A_0^{(i)} = 0$ (this justifies the validity of the scalar field equation used above). Thus $D_0 Z = \dot{Z}$. For our diagonal configurations, we have

$$\begin{aligned} F^{(1)} &= \frac{\pi i}{k} \left(\bar{Z} \dot{Z} - \dot{Z} Z \right) \\ F^{(2)} &= -\frac{\pi i}{k} \left(Z \dot{\bar{Z}} - \dot{\bar{Z}} \bar{Z} \right), \end{aligned} \tag{2.7}$$

from which we conclude

$$F^{(1)} = F^{(2)} = \frac{2\pi}{k} \text{diag}(|z_1|^2, |z_2|^2, \dots, |z_N|^2) \equiv 2\pi \text{diag}(n_1, n_2, \dots, n_N), \quad n_i \in \mathbb{Z}_+, \tag{2.8}$$

where quantization of the gauge field flux over the 2-sphere was used in the last step, implying that

$$|z_i|^2 = kn_i. \tag{2.9}$$

That is, $|z_i|^2$ is an (integer) multiple of the level k . From the above equation we learn that the quantized fluxes of the gauge fields n_i are a collection of *non-negative integers*. Acting with the permutation group S_N , we can arrange them such that $n_1 \geq n_2 \geq \dots \geq n_N$. We would like to stress that, being the eigenvalues of the matrix F , n_i are all gauge invariant quantities.

For the diagonal configurations specified by the set of fluxes $\{n_i\}$ the energy of the system is given by

$$H = \frac{1}{16\pi} \int_{S^2} d^2\Omega \text{Tr} (\bar{Z} Z + Z \bar{Z}) = \frac{k}{2} \sum_{i=1}^N n_i. \tag{2.10}$$

Note that the energy is also related to the total flux of the gauge fields over the two sphere, $H = \frac{k}{8\pi} (\text{Tr} F^{(1)} + \text{Tr} F^{(2)})$. One may also work out the angular momentum associated to the $u(1)$ rotation of the Z

$$J = \frac{1}{4\pi} \int d^2\Omega \text{Tr} (\bar{Z} D_0 Z - \overline{D_0 Z Z}) = \sum_{i=1}^N |z_i|^2 = k \sum_{i=1}^N n_i. \tag{2.11}$$

It is readily seen that $H = J/2$, which is the BPS condition.

Supersymmetry. So far we have argued that being in the half-BPS sector demands turning on an $SO(3)$ invariant mode of only one of the four complex scalars Z^A . Here we show that this is indeed enough for being half-BPS. To see this consider the supersymmetry variations for the fields in ABJM theories defined on $R \times S^2$ written in [17]. The amount of supersymmetry preserved by any bosonic configuration is obtained by solving

$$\delta\psi_{Bd} = \gamma^\mu D_\mu Z_d^A \epsilon_{AB} + f^{abc}{}_d Z_a^C Z_b^A \bar{Z}_{C\bar{c}} \epsilon_{AB} + f^{abc}{}_d Z_a^C Z_b^D \bar{Z}_{B\bar{c}} \epsilon_{CD} - \frac{1}{3} Z_d^A \gamma^\mu \nabla_\mu \epsilon_{AB} = 0.$$

In the above a, b, \dots are denoting the $u(N) \times u(N)$ bi-fundamental indices, $f^{abc}{}_d$ are the structure constants built from the $u(N)$ symmetric traceless and totally anti-symmetric three tensors [19] and ϵ_{AB} are fermions on $R \times S^2$ as well as being in the **6** of $SU(4)$ R-symmetry (denoted by A, B indices). In particular note that $\nabla_\mu \epsilon = \gamma_\mu \epsilon/2$, and $\gamma_0 \epsilon = i\epsilon$

and therefore, there are 12 independent ϵ 's.⁵ Since there is only one scalar $Z^1 = Z$ turned on the above reduces to

$$\delta\psi_{Bd} = \left(\gamma^\mu D_\mu Z_d - \frac{1}{2} Z_d \right) \epsilon_{1B} = 0 . \tag{2.12}$$

Any half-BPS configuration requires the above to vanish for any ϵ_{1B} (the other components of ϵ do not appear in (2.12) they can be chosen freely). This is only true if $D_\theta Z = D_\phi Z = 0$ and $D_0 Z = -(i/2) Z$. We note, however, that just checking the $\delta\psi = 0$ condition is not enough and one should make sure that all the equations of motion are also satisfied. For the Z field this is trivial, but not for the gauge fields (note that in our Chern-Simons theory the gauge fields are non-propagating). In particular, on top of (2.12), (2.4) should also be added, yielding to $F_{t\theta}^{(i)} = F_{t\phi}^{(i)} = 0$ and $F_{\theta\phi}^{(1)} = \frac{\pi}{k} \sin\theta \bar{Z} Z$, $F^{(2)} = \frac{\pi}{k} \sin\theta Z \bar{Z}$. These equations may be solved in the $A_\mu^{(1)} = A_\mu^{(2)}$ gauge and in the gauge where Z is diagonal; as was done in the previous section.

After discussing the supersymmetry condition let us also discuss the classification half-BPS states by the relevant superalgebras. The $\mathcal{N} = 8$ 3d theory is invariant under the 3d superconformal $Osp(4^*|8)$ superalgebra, which has 32 real supercharges and is the superisometry of $AdS_4 \times S^7$ geometry. This algebra has a number of sub-algebras with 16 supercharges which has been listed in [16]. The ABJM theory for generic k , on the other hand is an $\mathcal{N} = 6$ superconformal theory and its superalgebra is $Osp(4^*|6)$, which has 24 real supercharges.

The half-BPS sector we have been studying, which involves only one of the four complex scalars of the theory, is invariant under $SU(2|4)$ for the $\mathcal{N} = 8$ case (related to $k = 1, 2$ ABJM theories) and $SU(2|3)$ for the $\mathcal{N} = 6$ (generic k ABJM theory). That is, they fall into singlet representations of the above-mentioned half-BPS superalgebras. In either cases, the $SU(2)$ invariance is enforced in our construction by demanding invariance under the $SO(3)$ isometries of the S^2 in the radial quantization. The $SU(4)$ or $SU(3)$ invariance is made manifest in exciting only one of the four complex scalars. The generator of the $U(1)$ in these superalgebras is related to $\Delta - J/2$ in the CFT side and hence its invariance is enforced by imposing the BPS condition.

2.2 Construction of half-BPS states

In the previous section, we discussed the classical moduli space of half-BPS configurations consistent with the appropriate bosonic symmetries. We will now construct gauge invariant operators carrying the right charges corresponding to these classical configurations. By the operator-state correspondence, these will describe the half-BPS states in ABJM theories.

Such operators can only involve a single scalar matrix Z . Since this transforms in the bi-fundamental representation of the $U(N) \times U(N)$ gauge group, the trace over its matrix indices will not generate a gauge invariant operator. As already mentioned in [1], we can construct local gauge invariant operators using monopole or 't Hooft operators [8]. It is

⁵Note that besides the supersymmetry transformations generated by the ϵ_{AB} above, there are 12 extra fermionic transformations which all together form the $Osp(4^*|6)$ superalgebra [6]. Thus, half BPS states should preserve all ϵ_{AB} defined above. This is what we will prove below.

this part of the operator that will carry the information about the magnetic fluxes turned on in the classical configurations.

Before moving to explicit construction of the relevant monopole operators to our Chern-Simons theory, we note that turning on fluxes of the gauge fields, n_i will generically break the $u(N) \times u(N)$ gauge symmetry to $u(1)^N \times u(1)^N$. Let us denote the generators of this remaining Abelian subgroup by T_i^1 and T_i^2 where $i = 1, 2, \dots, N$. Under $u(1)^N \times u(1)^N$ transformations, $u(1)^N$ which is generated by $T_i^1 + T_i^2$, Z remains invariant and under those generated by $t_i \equiv T_i^1 - T_i^2$, Z rotates by a phase. In the notations of ABJM $U(1)_b$ is the part of the gauge symmetry which is generated by $t = \sum_{i=1}^N t_i$. The fluxes $\{n_i\}$ are then charges of t_i . We define the “total flux” J/k as

$$J = k \sum_{i=1}^N n_i. \tag{2.13}$$

J/k , is hence the flux corresponding to the $U(1)_b$ [1].

To illustrate the idea behind the construction of these operators, let us consider the $U(1) \times U(1)$ theory first. Working in the gauge $A^{(1)} = A^{(2)} = A$ with $A = \frac{n}{2}(1 - \cos \theta)d\phi$, the only left gauge transformations are the time dependent ones acting on A_0 as

$$A_0^1 \rightarrow A_0^1 + \partial_0 \lambda, \quad A_0^2 \rightarrow A_0^2 + \partial_0 \lambda.$$

The monopole operator is defined as the imaginary exponential of the integral of the Chern-Simons form over the 2-sphere and integrating time from $t \rightarrow -\infty$ to a value t . Since the gauge field carries n units of flux, such operator is characterized by (k, n, t) :⁶

$$W_{\text{monopole}}(n; t) \equiv e^{-i \frac{k}{4\pi} \int_{-\infty}^t dt A_0 \int d^2 \Omega F_{\theta\phi}} = e^{ikn \int_{-\infty}^t dt A_0}. \tag{2.14}$$

From now on, let us focus on the $\mathcal{N} = 8$ theory ($k = 1$). Assuming that all gauge transformations vanish as $t \rightarrow -\infty$, i.e. $\lambda(t \rightarrow -\infty) = 0$, we conclude the monopole operator transforms as $W_{\text{monopole}} \rightarrow e^{-in\lambda} W_{\text{monopole}}$ under a gauge transformation. Recalling that under the same gauge transformation $Z^n \rightarrow e^{+in\lambda} Z^n$, we conclude that $W(n; t) Z^n(t)$ is a gauge invariant operator. This operator has R-charge n and conformal dimension $n/2$. Since $W(n; t) Z^n(t)$ involves an integral over the local gauge field A_0 , it is a non-local operator. However, as discussed in [1] and as it can be seen from our arguments above, this is not the case. This is because the non-local part of this operator can be absorbed by (finite) gauge transformations. Notice also that this is the only gauge invariant operator carrying these charges one can build for this gauge group involving a single bi-fundamental matter field. This agrees with the classical moduli space analysis above. Equivalently, there is a one-to-one correspondence between the half-BPS operator and the magnetic flux n that characterizes the charges carried by the operator.

Let us extend this construction to the $U(N) \times U(N)$ theory, where we already know the magnetic fluxes are specified by N integers, and not just one as in the Abelian case.

⁶Although we are performing the computations in a gauge in which the background gauge field has only non-zero A_ϕ component, from (2.14) it is seen that our discussions go through for a generic gauge transformation λ and not just for those which are only time dependent.

In the general case, turning on the fluxes $\{n_i\}$ generically breaks the gauge group to $U(1)^N \times U(1)^N$. The individual eigenvalues z_i rotate under the $U(1)$ rotation generated by t_i . This suggests generalizing the above construction for each pair of unbroken $U(1) \times U(1)$ gauge group factors.

Proceeding as if we have N copies of the $U(1) \times U(1)$ theory and with a given set of fluxes $\{n_i\}$ the most general operator which is invariant under $U(1)^N \times U(1)^N$ and involves the monopole operators is hence⁷

$$\mathcal{O}_{\{n_i\}} = \prod_{i=1}^N W(n_i; t) z_i^{n_i}(t). \tag{2.15}$$

The above operator, which is written in a specific gauge, is in fact also invariant under the full $U(N) \times U(N)$ gauge symmetry. To see this, let us recall that a Wilson line (monopole) operator in a $U(N)$ gauge theory:

$$\mathcal{O}(t_i, t_f) = P \left(e^{i \int_{t_i}^{t_f} dt A_0} \right), \tag{2.16}$$

transform under $U(N)$ gauge transformations as

$$\mathcal{O} \longrightarrow U(t_i) \mathcal{O} U(t_f)^{-1}. \tag{2.17}$$

Thus, if we take $U(-\infty) = \mathbb{1}$, the corresponding operator

$$\mathcal{O}(t) = P \left(e^{i \int_{-\infty}^t dt A_0} \right) \tag{2.18}$$

is in the anti-fundamental of the $U(N)$ gauge group.

Let us now consider our $U(N) \times U(N)$ gauge theory. For each gauge group we can construct a 't Hooft operator which is in the (anti)fundamental of either of the gauge groups. Explicitly consider

$$\mathcal{O}_1 = P \left(e^{i \int_{-\infty}^t dt A_0^{(1)}} \right), \tag{2.19}$$

$$\mathcal{O}_2 = P \left(e^{-i \int_{-\infty}^t dt A_0^{(2)}} \right). \tag{2.20}$$

Clearly, \mathcal{O}_1 is in the anti-fundamental of the first $U(N)$ group whereas \mathcal{O}_2 is in the fundamental of the second $U(N)$. Since Z transforms in the bi-fundamental (N, \bar{N}) , we conclude that the operator $\mathcal{O}_1 Z \mathcal{O}_2$ is gauge invariant.

The previous construction has no relation to the individual fluxes $\{n_i\}$. To introduce the dependence on these quantum numbers, let us return to the matrix $F^{(a)}$ $a = 1, 2$. One may use the $U(N)$ large gauge (global) transformations to bring both matrices to a diagonal form:

$$F^{(a)} |i\rangle_{(a)} = 2\pi n_i^{(a)} |i\rangle_{(a)} \tag{2.21}$$

⁷Note that due to the $SO(3)$ invariance in the half-BPS sector one may simply reduce the theory on the S^2 and remain with a $0 + 1$ dimensional quantum mechanical system. The half-BPS operators are hence operators in this theory and have only time dependence.

We can now build projector operators:

$$\mathcal{P}_i^{(a)} = |i\rangle_{(a)}\langle i|_{(a)}. \tag{2.22}$$

By construction, each of these projectors transforms in the adjoint under a gauge transformation of the a^{th} $U(N)$ gauge group. Thus, the set of states

$$\mathcal{Q}_{ij} = \mathcal{O}_1 \mathcal{P}_i^1 Z \mathcal{P}_j^2 \mathcal{O}_2, \tag{2.23}$$

is gauge invariant.

To sum up, the product of the traces of these operators raised to any integer would be a half-BPS gauge invariant operator. However, due to the equations of motion for the gauge fields $F^{(1)} = F^{(2)}$. Thus, both basis are equal, $|i\rangle_{(1)} = |i\rangle_{(2)}$, and we can drop the dependence on this index. Similarly $Z_{ij} = \langle i|_{(1)} Z |j\rangle_{(2)}$ is also diagonal in this same basis.

Denoting by $\mathcal{Q} = \mathcal{O}_1 \mathcal{O}_2$ the operator transforming in the $(\bar{\mathbf{N}}, \mathbf{N})$ representation of $U(N) \times U(N)$ (using both 't Hooft operators defined above), we can write our gauge invariant operators as

$$\mathcal{Z}_i \equiv \mathcal{Q} \mathcal{P}_i Z. \tag{2.24}$$

We can finally write the operators $\mathcal{O}_{\{n_i\}}$ in terms of the \mathcal{Z}_i as

$$\mathcal{O}_{\{n_i\}} = \prod_{i=1}^N \mathcal{Z}_i^{n_i}. \tag{2.25}$$

Note that $\{n_i\}$ are also gauge invariant quantities.

The set of operators $\mathcal{O}_{\{n_i\}}$ satisfying $\sum_{i=1}^N n_i = J$ correspond to half-BPS operators with R-charge J . This explicit construction establishes a one-to-one correspondence between this class of half-BPS operators and the set of Young tableaux with J boxes and at most N rows: $\mathcal{O}_{\{n_i\}}$ corresponds to a Young tableau which has n_i number of boxes in the i^{th} row. The fact that such Young tableau do completely characterize the space of vacua of the plane wave Matrix Model strongly suggests that the operators constructed here are complete.⁸

For $k \neq 1$ one can check that all the above arguments go through and one needs to simply replace n_i in (2.25) by kn_i . Thus, in this case, all half-BPS operators are still completely specified by the set of $\{n_i\}$.

3 2d fermion picture

The half-BPS sector of $\mathcal{N} = 4$ SYM is described by a matrix model whose degrees of freedom correspond to free fermions in a one dimensional harmonic potential or equivalently $2d$ fermions in the lowest Landau Level. It is natural to wonder whether our $d = 3$ SCFTs have a similar fermionic description for their half-BPS sectors.

⁸The correspondence to the Matrix Model is known to be exact in the limit $N \rightarrow \infty$, in which the number of rows (rank of the gauge group) is not fixed.

Let us assume we can decouple this sector in the full theory and consider the corresponding partition function. Due to the $SO(3)$ invariance we may do the reduction on the sphere and remain with a $0 + 1$ one complex matrix model in the bi-fundamental of $u(N) \times u(N)$. As previously discussed, we can use the $U(N) \times U(N)$ gauge symmetry to diagonalize Z and work with its eigenvalues z_i (2.6) as degrees of freedom. The remaining $U(1)^N$ gauge symmetry can be used to remove the phases of z_i and make them all positive real values, which will be denoted by r_i . To fix the gauge in which Z is diagonal, however, we need to include the Jacobian of these gauge transformations into the measure of the path integral. Being in the bi-fundamental the procedure is a bit different from the one discussed for adjoint scalars in $4d$ SYM. This has been carried out in the context of complex matrix models in [20, 21] and here we sketch the argument.

To compute the measure factor, we recall the form of the gauge transformations on Z , $Z \rightarrow UZV^{-1}$. Under the infinitesimal gauge transformations

$$U \simeq 1 + i(\lambda + \rho), \quad V^{-1} \simeq 1 - i(\lambda - \rho), \quad (3.1)$$

where λ and ρ are $N \times N$ hermitian matrices, elements of $u(N)$ algebra, we have

$$\delta Z = i[\lambda, Z] + i\{\rho, Z\} . \quad (3.2)$$

Next let us assume that Z 's are diagonal:

$$Z = \text{diag}(z_1, z_2, \dots, z_N) \quad (3.3)$$

and choose the T^k basis for the $N \times N$ gauge transformations, where the elements of T^k $N \times N$ matrices are

$$(T^k)_{ij} = \delta_{i,j+k}, \quad i + N \equiv i . \quad (3.4)$$

We then have

$$\left[T^k, Z \right] = (z_i - z_{i+k})\delta_{i,j+k}, \quad \left\{ T^k, Z \right\} = (z_i + z_{i+k})\delta_{i,j+k} . \quad (3.5)$$

As mentioned earlier, after diagonalization of Z we are still left with the residual $u(1)^N$ gauge symmetry and one may use that to bring z_i to positive real valued r_i , $r_i = |z_i|^2$.⁹

The Jacobian of these gauge transformations equals

$$\begin{aligned} J &\equiv \left| \frac{\delta Z}{\delta \lambda} \cdot \frac{\delta Z}{\delta \rho} \right|^2 \\ &= \left| \prod_{k=1}^N \prod_{i=1}^k (z_i - z_{i+k}) \cdot (z_i + z_{i+k}) \right|^2 = \left| \prod_{i>j} (z_i^2 - z_j^2) \right|^2, \end{aligned} \quad (3.6)$$

The measure of the path integral in the half-BPS sector involves $DZD\bar{Z}$ after fixing the gauge and in the basis where Z is diagonal it becomes $\prod_i dz_i d\bar{z}_i \prod_{i>j} (z_i^2 - z_j^2)^2$. The residual $u(1)^N$ symmetry should now be implemented. This will not change the Jacobian (3.6)

⁹To be more precise, in the presence of the Chern-Simons terms we are still left with the over-all $u(1)$, the $u(1)_b$ which rephases all z_i simultaneously and hence all the z_i have the same phase.

and its effect is to render z_i real positive and reducing $dz_i d\bar{z}_i$ piece to $r_i dr_i$. In summary, the gauge fixed measure is $\prod_i r_i dr_i \prod_{i>j} (r_i^2 - r_j^2)^2$ [20].

In analogy with the half-BPS sector of an $\mathcal{N} = 4$ U(N) SYM theory (e.g. see [10, 22, 23]), one can then rewrite the partition function of the ABJM theory in the half-BPS sector as

$$Z|_{half-BPS} = e^{-F} = \int [DA_1 DA_2 DZ D\bar{Z}]_{half-BPS} e^{-S_{1/2 BPS}} = \langle \Psi | \Psi \rangle \quad (3.7)$$

where $S_{1/2 BPS}$ is (2.1) but reduced on S^2 and $|\Psi\rangle$ is the wavefunction for the *vacuum state* of a system of N $2d$ fermions in a harmonic oscillator potential. In the above by $[DA_1 DA_2 DZ D\bar{Z}]_{half-BPS} \equiv DM$ we mean the part of the measure which is allowed by the half-BPS condition. In other words, we are assuming that the half-BPS sector is a closed sector of the theory and is protected, even quantum mechanically, by supersymmetry. In particular, in DM we do not allow for Z and A configurations which have non-vanishing $D_\theta Z$, $D_\phi Z$, $F_{\theta t}^{(i)}$ or $F_{\phi t}^{(i)}$. Moreover, DM has a $\delta(2F_{\theta\phi}^{(i)} / \sin\theta - J_0^{(i)})$ factor. Therefore, what we are computing is effectively the partition function of a $0 + 1$ dimensional one complex matrix model which is exactly equal to the partition function of N $2d$ fermions in a harmonic oscillator potential. The residual $u(1)^N$ symmetry, however, amounts to setting the angular momentum of each of these oscillators on the $2d$ plane equal to zero [20]. Explicitly, each of these fermions should satisfy the Schrodinger equation

$$-\frac{1}{r_i} \partial_{r_i} (r_i \partial_{r_i} \Psi_i) + r_i^2 \Psi_i = 2(2n_i + 1) \Psi_i \quad (3.8)$$

The frequency of this system, as is seen from (2.10), is $k/4$. Note that working in the zero angular momentum sector, the energy (once the zero point energy 2 is subtracted) is an *even* multiple of the frequency and hence on the right-hand-side of (3.8) we have 4 times an integer. The state $|\Psi\rangle$ is then obtained from the Slater determinant of Ψ_i 's, which is leading to the measure factor times a Gaussian with width one. In this picture the factor $r_i dr_i$ is naturally related to the fact that we are working with $2d$ fermions.

It is worth noting that despite the similarities there are important differences with the $\mathcal{N} = 4$ SYM case:

- In the $\mathcal{N} = 4$ case, the half-BPS sector can be described through a system of $2d$ fermions in the presence of a constant magnetic field in the Lowest Landau Level.¹⁰ In our case, degrees of freedom can be interpreted as $2d$ fermions in an harmonic potential in states of vanishing angular momentum. Thus, there is no relation to the quantum Hall system nor the Laughlin wave function. On the other hand, the zero $2d$ angular momentum condition can be related to a one dimensional “half harmonic oscillator potential” (restricted to move in the $x > 0$ region). However, the latter will not produce the extra r_i factor in the measure.
- In the ABJM theory, there is a non-trivial flux over the S^2 coming from the insertion of the 't Hooft loop operators (2.15). This should be contrasted with the $\mathcal{N} = 4$ on

¹⁰This was related to the fact that in the SYM side half-BPS states are holomorphic functions of one of the three scalars complex scalars of the theory [10].

$R \times S^3$ (*cf.* footnote 4). This is implemented by performing the path integral around the configuration with these background fluxes (given in (2.8)). This also leads to the appearance of $k/4$ as the frequency of the $2d$ harmonic oscillator.

4 Half-BPS states in dual descriptions

The ABJM theory (at level $k = 1$) is dual to M-theory on $AdS_4 \times S^7$. We have two different available descriptions for the latter: the $11d$ supergravity in asymptotically $AdS_4 \times S^7$ backgrounds and the plane-wave matrix model [7]. In the following, we will match the half-BPS operators constructed in previous sections with the description of these states in these other two formulations of the same system. This will provide a check of our operator construction in the $N \rightarrow \infty$ limit.

4.1 Plane-wave matrix theory perspective

The following discussion is strongly based on the results obtained in [12, 15, 24, 25]. We review them here for completeness to establish a precise relation with the half-BPS operators defined before.

The plane-wave matrix model [7] is a $0+1$ dimensional $U(J)$ supersymmetric quantum mechanics involving nine scalars X^A and their fermionic counterparts, all in the $J \times J$ hermitian representation of the $U(J)$ gauge group. The set of scalars is divided into two groups X^a , $a = 1, 2, \dots, 6$ and X^i , $i = 1, 2, 3$. Physical states lie in representations of $SU(2|4)$ comprised of finite collections of representations of the bosonic subalgebra $SO(6) \times SO(3) \times U(1)_H$.

It is known that this matrix model has a discrete set of classical half-BPS vacua which are interpreted as fuzzy M2-brane spheres. The half-BPS condition implies $X^a = 0$ and the kinetic terms to vanish and we hence remain with

$$H = \frac{R_-}{8} \text{Tr} \left(i\epsilon^{ijk} [X^i, X^j] + \frac{\mu}{2R_-} X^k \right)^2 \tag{4.1}$$

(R_- is the arbitrary energy scale of the theory and μ/R_- is the only dimensionless parameter of this theory. These would be irrelevant to our discussion of half-BPS states.) Zero energy configurations are hence solutions to

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{4.2}$$

where $X^i = \frac{\mu}{2R_-} J_i$. Thus, all classical vacua are labeled by J dimensional reducible representations of $SU(2)$. The latter are determined by a set of m irreducible representations of size N_i appearing n_i times in the decomposition of the initial reducible representation so that

$$J = \sum_{i=1}^m n_i N_i. \tag{4.3}$$

Clearly, the set of all classical vacua is equivalent to the problem of partition of an integer J into non-negative integers [25], or equivalently to the set of Young tableaux with J boxes. This is exactly the same set characterizing our proposed half-BPS operators in

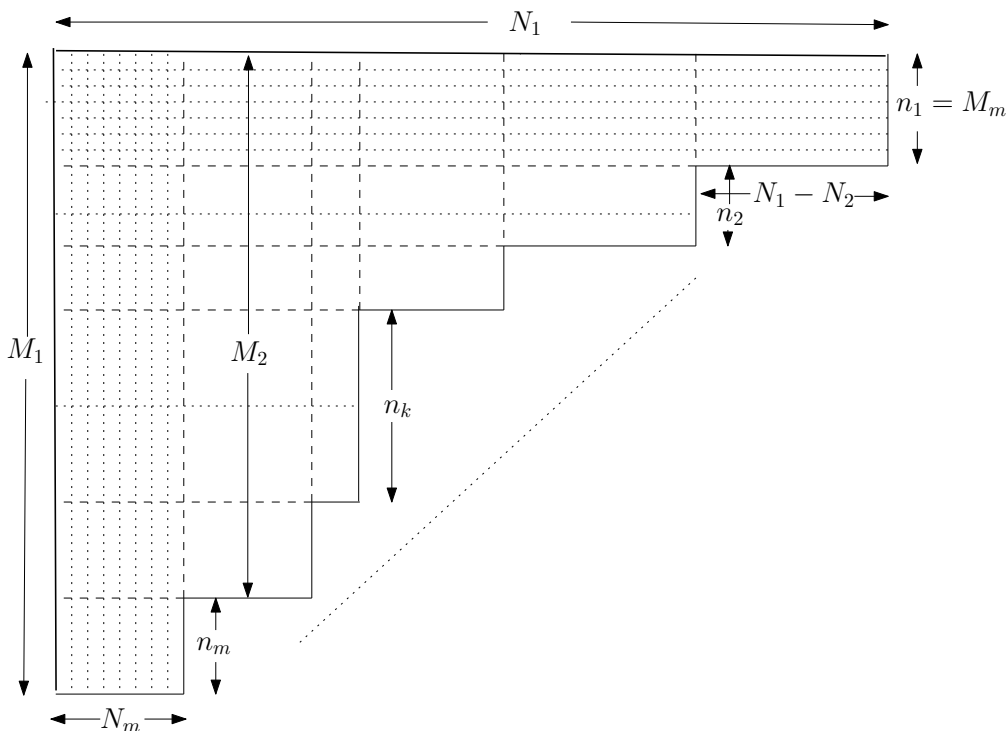


Figure 1. A given Young tableau can have interpretation in terms of a collection of n_i spherical M2-branes of radius N_i , $i = 1, 2, \dots, m$ or alternatively in terms of m_k spherical M5-branes the radius of which is given by M_k , $k = 1, 2, \dots, m$. As is seen from (4.4), $m_1 = N_m$, $m_2 = N_{m+1} - N_m, \dots, m_m = N_1 - N_2$.

the ABJM theories. Since it is known that these states are exact quantum mechanical vacua, and the size of the representation corresponds to the units of light-cone momentum carried by the state in its DLCQ interpretation, we can conclude our matching goes beyond the classical identification.

The microscopic interpretation in terms of (quantized) spherical M2-branes and M5-branes is similar to the one encountered in the half-BPS sector of $\mathcal{N} = 4$ SYM. Spherical M2-branes correspond to dual giant gravitons whose size is proportional to the size of the irreducible representation N_i ; n_i stands for the number of dual giants having the same size. In terms of the Young tableau description, we can always order the sizes of the irreducible representations so that $N_i > N_j$ for $i < j$. In this way, a given Young tableau has n_i rows of length N_i , with the total number of rows $\sum_i n_i$ being the total number of M2-brane giants.

As argued in [25], the same Young tableau and set of labels can have an interpretation in terms of (quantized) spherical M5-brane giants, as a collection of m M5-branes the fourth power of the radius of which is proportional to the amount of the DLCQ light-cone momentum they carry, M_k , and there are m_k five branes of a given size.¹¹ The M2-brane

¹¹Note that both size and number of giants are “classical” notions and are not good quantum numbers in an interacting theory. Even though the M2 or M5 -giant interpretation is not appropriate one in finite J , finite N_i or finite M_i , where the M2 or M5 brane theory becomes strongly coupled, labeling vacua by these quantum numbers is still meaningful in such cases.

and M5-brane parameters, as depicted in figure 1, are related as:

$$M_k = \sum_{i=1}^{m-k+1} n_i, \quad m_k = N_{m-k+1} - N_{m-k+2}, \quad k = 1, 2, \dots, m, \quad N_{m+1} = 0. \quad (4.4)$$

One can easily check that $\sum_{i=1}^m m_i M_i = \sum_{i=1}^m n_i N_i = J$. The M5-brane description becomes a good one (weakly coupled) when the M_i are large, while the M2 description is a good one when N_i are large.

Modulo the caveats associated with interpreting these states geometrically as bound states of spherical M2 and M5-brane giants, we can definitely establish a one-to-one correspondence between any set of fluxes $\{n_i\}$ determining our half-BPS operators with the set of dimensions of the irreducible representations characterizing the plane-wave Matrix model vacua. Moreover, the ABJM theory (at $k = 1$) is invariant under the $Osp(4^*|8)$ superalgebra. The latter has several half-BPS sub-superalgebras [16]. The one relevant after taking the Penrose limit is $SU(2|4)$. That is precisely the one corresponding to the $SU(2)$ invariant half-BPS states of the ABJM theory. This closes the argument regarding the comparison between the ABJM $SU(2)$ half-BPS spectrum studied in section 2 and the plane-wave matrix model. Note, however, that in the ABJM theory the corresponding Young tableau has a maximum number of rows N . In the case of the plane-wave matrix model, as we are dealing with M-theory on the Penrose limit of $AdS_4 \times S^7$, N has been sent to infinity.

Before ending this section, we would also like to comment on the $SU(2)$ -invariant half-BPS sector of the ABJM theory with $k \neq 1$. Recalling the process of taking the Penrose limit and that for generic k the ABJM theory is dual to M-theory on $AdS_4 \times S^7/Z_k$, this half-BPS sector corresponds to making the shift $R_- \rightarrow R_-/k$. Equivalently, we can replace the size of the matrices by kJ , while only restricting them to fall into reducible $SU(2)$ representations with block sizes being multiples of k , i.e. for N_i being integer multiples of k .

4.2 11d supergravity perspective

The following discussion is based on [2, 13, 15]. We include it here for completeness and to make the emergence of Young tableau from the classical moduli space of supergravity configurations and its relation to the Young tableau appearing in our operator construction more explicit. A similar connection, using the plane-wave matrix model vacua and supergravity has been discussed in [13, 15].

The classical moduli space of half-BPS configurations preserving $SO(6) \times SO(3) \times U(1)$ in 11d supergravity was worked out in [2] and we very briefly review them here. These symmetries reduce the functional dependence of all metric and flux components to a three dimensional dependence described by a set of coordinates $\{y, x_1, x_2\}$. Any solution belonging to this moduli space is determined by a scalar function $D(y, x_1, x_2)$ satisfying the Toda equation

$$(\partial_1^2 + \partial_2^2)D + \partial_y^2 e^D = 0. \quad (4.5)$$

It was also pointed out that for any translationally or rotationally invariant configuration, the Toda equation could be mapped through a non-linear change of variables to a linear 3d Laplace equation. For our purposes of establishing a dictionary between our

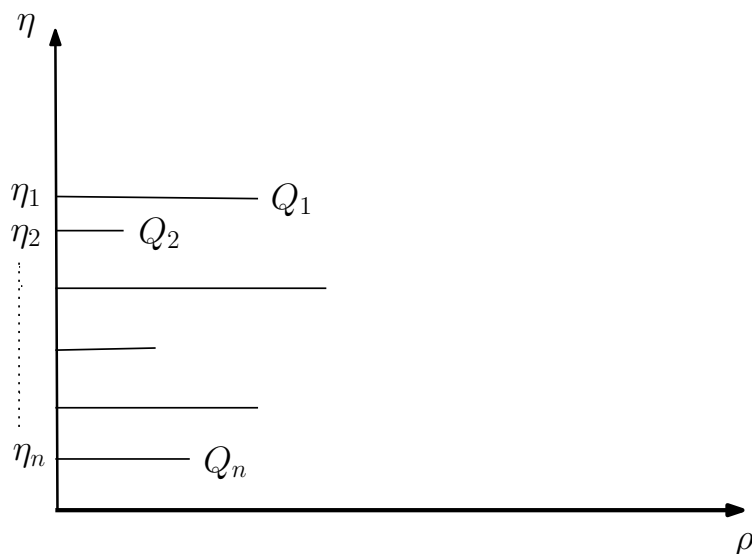


Figure 2. A generic half-BPS deformation of the 11d plane-wave is specified with a distribution of conducting disks with charge Q_i located at η_i on the $\eta - \rho$ plane.

proposed half-BPS operators to supergravity configurations, it will be enough to restrict ourselves to translationally invariant solutions. These were extensively studied in [13–15]. In such situation, the Toda equation reduces to

$$\partial_2^2 D + \partial_y^2 e^D = 0. \tag{4.6}$$

Using the implicit change of coordinates and variables

$$e^D = \rho^2, \quad y = \rho \partial_\rho V(\rho, \eta), \quad x^2 = \partial_\eta V(y, \eta), \tag{4.7}$$

the Toda equation is mapped to the axisymmetric 3d Laplace equation

$$\frac{1}{\rho} \partial_\rho (\rho \partial_\rho V) + \partial_\eta^2 V = 0. \tag{4.8}$$

This is a problem in electrostatics with potential V which can be fully determined once we impose a set of boundary conditions that makes these half-BPS configurations regular. It was shown in [13] that this mathematical problem is fixed by specifying the locations η_i and the charges Q_i carried by a discrete set of conducting disks (their sizes are related to the charges they carry).

All these solutions will be deformations of the maximally supersymmetric 11d plane-wave background, so let us consider this solution first. The electrostatic potential is

$$V_b = \rho^2 \eta - \frac{2}{3} \eta^3. \tag{4.9}$$

This fixes the transformation between coordinates to be

$$y = 2\rho^2 \eta, \quad x^2 = \rho^2 - 2\eta^2, \tag{4.10}$$

whereas the 11d metric is

$$ds^2 = -4(4\eta^2 + \rho^2)dt^2 - 4dx^1 dt + 4(d\rho^2 + d\eta^2) + 4\rho^2 d\Omega_5^2 + 4\eta^2 d\Omega_2^2 . \quad (4.11)$$

Notice that both η and ρ are coordinates related to the radii of S^2 and S^5 . This point will be important in our analysis below.

Any excitation on top of this vacuum will be given by a distribution of conducting disks located at constant $\eta > 0$ (see figure 2).¹² The disks locations are given by positive η_i and their sizes/charges by ρ_i (*cf.* figure 2). The number of M2-branes (N_2) and M5-branes (N_5) can be computed in the supergravity approximation as flux integrals [13]:

$$N_2 = \frac{8Q_i}{\pi^2}, \quad (4.12)$$

$$N_5 = \frac{2d_i}{\pi}, \quad (4.13)$$

where Q_i is the charge of the i^{th} disk and $d_i = \eta_{i+1} - \eta_i$. We comment that in the above picture we have considered M2-branes and M5-branes as perturbations above the background plane-wave and hence it is meaningful to specify the size and number of branes.

With the above information we can readily map figure 2 into the Young diagram corresponding to the plane-wave matrix theory vacua. This has been depicted in figure 3 (a similar picture has been discussed in [14]). Note that we have chosen the vertical axis such that it directly measures Q_i rather than the ρ .

5 Discussion

We have studied and classified all the $SO(3)$ invariant states in the half-BPS sector of the $u(N) \times u(N)$ ABJM theory on $R \times S^2$. First, we determined the classical moduli space of such configurations and showed that in the sector with R-charge J it coincides with the solutions to the problem of partition of J into N non-negative integers. These integers correspond to the integral fluxes of the gauge fields on the S^2 where the theory is defined. Therefore, these states are in one-to-one correspondence with Young tableaux of J boxes and maximum number of rows N . Second, we constructed explicit gauge invariant BPS operators involving non-Abelian 't Hooft monopole operators.

We showed, through path integral considerations, that partition function of the ABJM theory in this sector is exactly matching the partition function of N $2d$ fermions in a harmonic oscillator potential the frequency of which is $k/4$ and where the fermions are restricted to move in the zero angular momentum sector. This should be contrasted with the fermionic picture corresponding to the half-BPS sector of $\mathcal{N} = 4$ SYM theory. It would be desirable to match our partition function arguments to the computations of supersymmetric indices [27] and semi-short operator counting [28] carried for the ABJM theory.

We argued that there is a precise correspondence between the half-BPS sectors of ABJM theory, plane-wave matrix theory and the 11d LLM bubbling geometries and all of

¹²Note that the (ρ, η) plane is actually a half-plane, since $\rho \geq 0$ and the background potential V_b fills the $\eta \leq 0$ region.

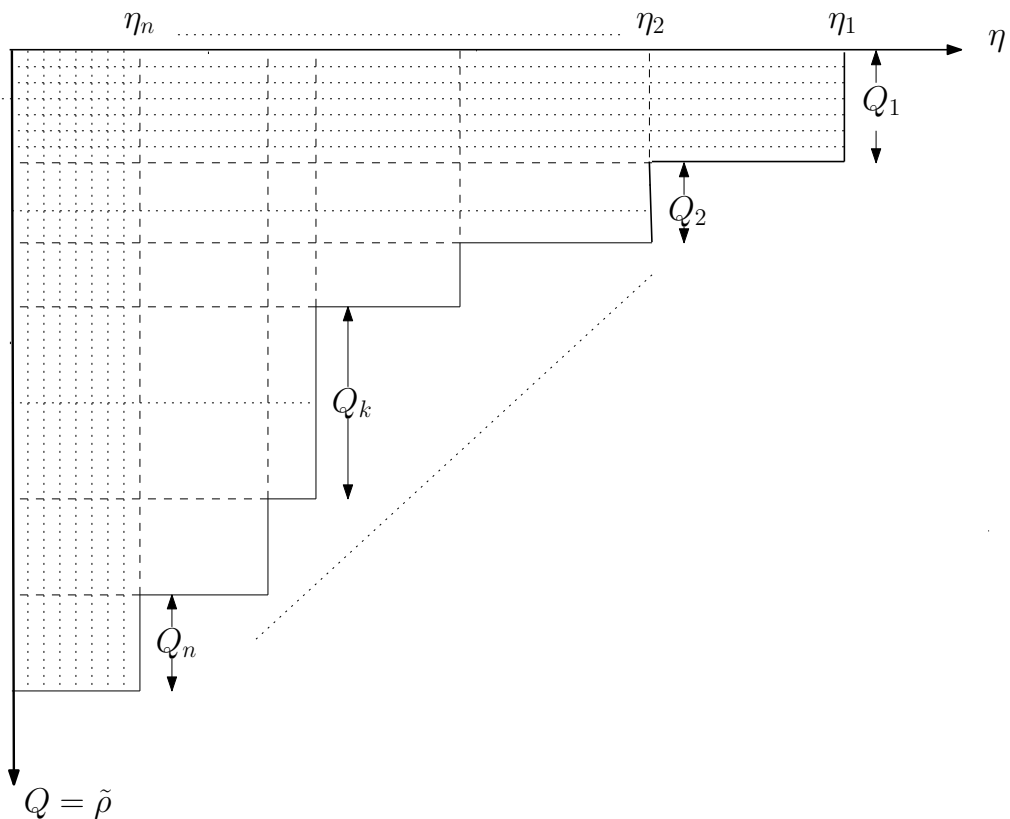


Figure 3. The connection between LLM $\eta - \rho$ plane and the Young diagram. The length of the rows are determining the size of M2-branes and the number of row of the same size, Q_i , is determining the number of M2-branes of a given size. Alternatively one can focus on the columns. The length of columns determine the size of M5-branes while the number of columns of given length, $\eta_{i+1} - \eta_i$, determines the number of M5-branes with that size. The total number of boxes in the Young diagram is hence $\sum_{i=1}^n \eta_i Q_i$ which is equal to the total R-charge of the system (above the background value).

which can be nicely encoded in terms of Young diagrams, e.g. see figure 3. This precise matching was, however, done for $N \rightarrow \infty$ case. This was due to the computational difficulties of constructing the 11d LLM bubbling geometries which are deformations of $AdS_4 \times S^7$ (or $AdS_7 \times S^4$), rather than the 11d plane-wave. One may use our results coming from ABJM theory as an additional guide to construct such solutions. In particular, our analysis of the measure and the 2d fermion picture (for $k = 1$ case) suggests that a similar fermionic picture, as we have in the 10d LLM geometries [2], should also be present in the 11d case.

As another related interesting question, for the half-BPS sector in type IIB on $AdS_5 \times S^5$, it was established that the singular half-BPS superstar supergravity configuration emerges as a coarse-grained description of the typical state in the Hilbert space describing N free fermions in the matrix model [29]. In M-theory, there is a similar singular half-BPS configuration, and it would be interesting to work out the dictionary between the gravity data describing the classical moduli space and the classical limit of the typical

quantum states belonging to this sector. The analysis of such “typical states” in the gauge theory side using the plane-wave matrix model was studied in [15]. If the wave functions for these states do not get renormalized, it should be possible to establish a connection between gauge theory and gravity. If there is renormalization, such connection may not be so apparent as for the $\mathcal{N} = 4$ story described in [29].

As argued the matrix theory and supergravity descriptions can be interpreted in terms of M2-branes or M5-branes. In the ABJM theory, being a $2 + 1$ dimensional field theory, the more natural interpretation is in terms of M2-branes. It is interesting to elaborate further on the M5-brane picture in the ABJM theory. One specific computation in this direction could be studying the spectrum of fluctuations of the theory around its half-BPS vacua. A similar analysis within the plane-wave matrix theory revealed [25] that among these fluctuations those which are protected by supersymmetry may be identified with the fluctuations of spherical M5-branes.

As argued in [16, 28] the $Osp(4^*|8)$ has other half-BPS superalgebras than $SU(2|4)$. It would be interesting to study the moduli space of half-BPS states which are invariant under these other half-BPS superalgebras. As a direct continuation of our half-BPS analysis one may also study and classify less BPS states. This problem has been considered e.g. in [18] and [27].

Acknowledgments

We would like to thank Hai Lin for his collaboration at the early stages of this work and David Berenstein, Juan Maldacena and Ofer Aharony for comments and discussions. M.M.Sh-J. would like to thank the Abdus Salam ICTP where a part of this work was carried out. The work of J.S. was partially supported by the Engineering and Physical Sciences Research Council [grant number EP/G007985/1]. This research was supported in part by the National Science Foundation under Grant No. NSF PHY05-51164. J.S. would like to thank the organizers of the KITP programme “Fundamentals of String Theory” for hospitality during the final stages of this project.

References

- [1] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, $N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[SPIRES](#)].
- [2] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [[hep-th/0409174](#)] [[SPIRES](#)].
- [3] J. Bagger and N. Lambert, *Modeling multiple M2’s*, *Phys. Rev. D* **75** (2007) 045020 [[hep-th/0611108](#)] [[SPIRES](#)]; *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev. D* **77** (2008) 065008 [[arXiv:0711.0955](#)] [[SPIRES](#)];
A. Gustavsson, *Algebraic structures on parallel M2-branes*, *Nucl. Phys. B* **811** (2009) 66 [[arXiv:0709.1260](#)] [[SPIRES](#)]; *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [[arXiv:0802.3456](#)] [[SPIRES](#)].

- [4] J.P. Gauntlett and J.B. Gutowski, *Constraining maximally supersymmetric membrane actions*, [arXiv:0804.3078](#) [SPIRES];
G. Papadopoulos, *M2-branes, 3-Lie algebras and Plucker relations*, *JHEP* **05** (2008) 054 [[arXiv:0804.2662](#)] [SPIRES].
- [5] M. Van Raamsdonk, *Comments on the Bagger-Lambert theory and multiple M2-branes*, *JHEP* **05** (2008) 105 [[arXiv:0803.3803](#)] [SPIRES].
- [6] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *Studies of the ABJM theory in a formulation with manifest SU(4) R-symmetry*, *JHEP* **09** (2008) 027 [[arXiv:0807.0880](#)] [SPIRES].
- [7] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from N = 4 super Yang-Mills*, *JHEP* **04** (2002) 013 [[hep-th/0202021](#)] [SPIRES].
- [8] V. Borokhov, A. Kapustin and X.-k. Wu, *Topological disorder operators in three-dimensional conformal field theory*, *JHEP* **11** (2002) 049 [[hep-th/0206054](#)] [SPIRES]; *Monopole operators and mirror symmetry in three dimensions*, *JHEP* **12** (2002) 044 [[hep-th/0207074](#)] [SPIRES];
N. Itzhaki, *Anyons, 't Hooft loops and a generalized connection in three dimensions*, *Phys. Rev. D* **67** (2003) 065008 [[hep-th/0211140](#)] [SPIRES].
- [9] Y. Imamura, *Monopole operators in N = 4 Chern-Simons theories and wrapped M2-branes*, [arXiv:0902.4173](#) [SPIRES]; *Charges and homologies in AdS₄/CFT₃*, [arXiv:0903.3095](#) [SPIRES].
- [10] A. Ghodsi, A.E. Mosaffa, O. Saremi and M.M. Sheikh-Jabbari, *LLL vs. LLM: half BPS sector of N = 4 SYM equals to quantum Hall system*, *Nucl. Phys. B* **729** (2005) 467 [[hep-th/0505129](#)] [SPIRES].
- [11] A. Shomer, *Penrose limit and DLCQ of string theory*, *Phys. Rev. D* **68** (2003) 086002 [[hep-th/0303055](#)] [SPIRES];
M.M. Sheikh-Jabbari, *Tiny graviton matrix theory: DLCQ of IIB plane-wave string theory, a conjecture*, *JHEP* **09** (2004) 017 [[hep-th/0406214](#)] [SPIRES].
- [12] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Matrix perturbation theory for M-theory on a PP-wave*, *JHEP* **05** (2002) 056 [[hep-th/0205185](#)] [SPIRES].
- [13] H. Lin and J.M. Maldacena, *Fivebranes from gauge theory*, *Phys. Rev. D* **74** (2006) 084014 [[hep-th/0509235](#)] [SPIRES].
- [14] H. Ling, A.R. Mohazab, H.-H. Shieh, G. van Anders and M. Van Raamsdonk, *Little string theory from a double-scaled matrix model*, *JHEP* **10** (2006) 018 [[hep-th/0606014](#)] [SPIRES].
- [15] H.-H. Shieh, G. van Anders and M. Van Raamsdonk, *Coarse-graining the Lin-Maldacena geometries*, *JHEP* **09** (2007) 059 [[arXiv:0705.4308](#)] [SPIRES].
- [16] E. D'Hoker, J. Estes, M. Gutperle, D. Krym and P. Sorba, *Half-BPS supergravity solutions and superalgebras*, *JHEP* **12** (2008) 047 [[arXiv:0810.1484](#)] [SPIRES].
- [17] M. Ali-Akbari, *3D CFT and multi M2-brane theory on R × S²*, *JHEP* **03** (2009) 148 [[arXiv:0902.2869](#)] [SPIRES].
- [18] D. Berenstein and D. Trancanelli, *Three-dimensional N = 6 SCFT's and their membrane dynamics*, *Phys. Rev. D* **78** (2008) 106009 [[arXiv:0808.2503](#)] [SPIRES].
- [19] J. Bagger and N. Lambert, *Three-algebras and N = 6 Chern-Simons gauge theories*, *Phys. Rev. D* **79** (2009) 025002 [[arXiv:0807.0163](#)] [SPIRES];

- M.M. Sheikh-Jabbari, *A new three-algebra representation for the $\mathcal{N} = 6$ $SU(N) \times SU(N)$ superconformal Chern-Simons theory*, *JHEP* **12** (2008) 111 [[arXiv:0810.3782](#)] [[SPIRES](#)].
- [20] M.R. Douglas et al., *A new hat for the $c = 1$ matrix model*, [hep-th/0307195](#) [[SPIRES](#)].
- [21] I.R. Klebanov, J.M. Maldacena and N. Seiberg, *Unitary and complex matrix models as $1 - D$ type 0 strings*, *Commun. Math. Phys.* **252** (2004) 275 [[hep-th/0309168](#)] [[SPIRES](#)].
- [22] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [[hep-th/0111222](#)] [[SPIRES](#)].
- [23] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [[hep-th/0403110](#)] [[SPIRES](#)].
- [24] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Protected multiplets of M-theory on a plane wave*, *JHEP* **09** (2002) 021 [[hep-th/0207050](#)] [[SPIRES](#)].
- [25] J.M. Maldacena, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Transverse fivebranes in matrix theory*, *JHEP* **01** (2003) 038 [[hep-th/0211139](#)] [[SPIRES](#)].
- [26] N. Drukker, J. Gomis and D. Young, *Vortex loop operators, M2-branes and holography*, *JHEP* **03** (2009) 004 [[arXiv:0810.4344](#)] [[SPIRES](#)].
- [27] J. Bhattacharya and S. Minwalla, *Superconformal indices for $\mathcal{N} = 6$ Chern Simons theories*, *JHEP* **01** (2009) 014 [[arXiv:0806.3251](#)] [[SPIRES](#)];
S. Kim, *The complete superconformal index for $\mathcal{N} = 6$ Chern-Simons theory*, *Nucl. Phys. B* **821** (2009) 241 [[arXiv:0903.4172](#)] [[SPIRES](#)].
- [28] F.A. Dolan, *On superconformal characters and partition functions in three dimensions*, [arXiv:0811.2740](#) [[SPIRES](#)].
- [29] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, *The library of Babel: on the origin of gravitational thermodynamics*, *JHEP* **12** (2005) 006 [[hep-th/0508023](#)] [[SPIRES](#)].